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Resolvent characterisation of generators of cosine functions and C_0-groups

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Abstract. The paper provides new characterisations of generators of cosine functions and C_0-groups on UMD spaces and their applications to some classical problems in cosine function theory. In particular, we show that on UMD spaces, generators of cosine functions and C_0-groups can be characterised by means of a complex inversion formula. This allows us to provide a strikingly elementary proof of Fattorini's result on square root reduction for cosine function generators on UMD spaces. Moreover, we give a cosine function analogue of McIntosh's characterisation of the boundedness of the H^∞ functional calculus for sectorial operators in terms of square function estimates. Another result says that the class of cosine function generators on a Hilbert space is exactly the class of operators which possess a dilation to a multiplication operator on a vector-valued L_2 space. Finally, we prove a cosine function analogue of the Gomilko-Feng-Shi characterisation of C_0-semigroup generators and apply it to answer in the affirmative a question by Fattorini on the growth bounds of perturbed cosine functions on Hilbert spaces.

1. Introduction

Recall that for a Banach space X, a strongly continuous function $C : \mathbb{R} \to \mathcal{L}(X)$ is called a cosine function if it satisfies d'Alembert's equation

$$C(t+s) + C(t-s) = 2C(t)C(s), \quad t, s \in \mathbb{R},$$

and $C(0) = I$. One can show that a cosine function is exponentially bounded with non-negative growth bound $\omega(C)$, i.e.,

$$0 \le \omega(C) := \inf \left\{ \omega \in \mathbb{R} : \sup_{t \ge 0} \|e^{-\omega t} C(t)\| < \infty \right\} < \infty.$$

Its generator is defined as the unique operator A such that

$$\lambda R(\lambda^2, A) = \int_0^\infty e^{-\lambda t} C(t) \mathrm{d}t, \quad \lambda > \omega(C). \tag{1.1}$$

Moreover, an operator A on X is the generator of a cosine function if and only if $(\omega, \infty) \subset \rho(A)$ for some $\omega \in \mathbb{R}$ and $(\cdot)R((\cdot)^2, A)$ is the Laplace transform of an operator-valued, strongly continuous function on $\mathbb{R}_+$.

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It is known that if A generates a cosine function on X, then the following two abstract Cauchy problems are well-posed in the sense of classical and mild solutions:

$$u''(t) = Au(t), \qquad t \geq 0, \;\; u'(0) = x, \;\; u(0) = y, \;\; x, y \in X, \tag{1.2}$$

$$u'(t) = Au(t), \qquad t \geq 0, \;\; u(0) = x, \;\; x \in X. \tag{1.3}$$

Furthermore, the classical result due to Fattorini [16] on square root reduction of cosine function generators implies that, under some additional assumptions on A and the underlying Banach space X, the following abstract Schrödinger equation

$$u'(t) = -i(-A)^{1/2}u(t), \qquad t \in \mathbb{R}, \;\; u(0) = x, \;\; x \in X, \tag{1.4}$$

is well-posed, too, i.e., $i(-A)^{1/2}$ generates a C_0-group U on X. In this case, we say that the cosine function C generated by A admits a *group decomposition*, since one can show that $C(t) = \frac{1}{2}(U(t) + U(-t))$ for every $t \in \mathbb{R}$. We refer the reader primarily to [2, Section 3.14–15] for the proofs of these results and to [28] for more information on fractional powers and functional calculus of sectorial operators.

In a series of papers [16–19], Fattorini developed cosine function theory mainly in connection with the Cauchy problems (1.2), (1.3) and (1.4).

There have been a number of recent papers exploring and improving Fattorini's results on relationships between cosine functions and C_0-(semi)groups associated with them. For instance, in [31], Haase proposed a new approach to the study of the group decomposition of uniformly bounded cosine functions on UMD spaces via the Phillips functional calculus, the transference principle and the theory of operator-valued Fourier multipliers; see also [29]. Another approach based on an inversion formula of Widder's type for a conjugate potential transform and the theory of boundary values of holomorphic semigroups was established in [9,37]. See also [8] for the group decomposition of cosine sequences.

The problem of the group decomposition of cosine functions is directly related to the characterisations of generators of cosine functions and C_0-groups, which were also the subject of intensive studies in recent years. In [42], Miana characterised cosine functions on Banach spaces in terms of a vector-valued cosine transform. An interesting characterisation by means of the numerical range can be found in [28, Corollary 7.4.8]. See also the characterisation in terms of boundary values of associated holomorphic semigroups from [37]. For characterisations of C_0-group generators on Hilbert spaces, we refer the reader to the result due to Haase [27], which completes the earlier studies in this direction from [7,40,47].

Although all the above-mentioned characterisations are obtained by different approaches, in a sense, the characterisations of C_0-group generators are incompatible with the characterisations of cosine function generators, especially if we think about the group decomposition of cosine functions.

In the first part of this paper, we provide characterisations of generators of cosine functions and C_0-groups on UMD spaces which, in particular, throw a new light on the phenomenon of the group decomposition of cosine functions, see Theorems

2.5 and 2.6. The proofs of these results are based on a regularity property which is common for the solutions of Cauchy's and d'Alembert's equation, see Lemma 3.1. In order to apply this property in the proofs of Theorems 2.5 and 2.6, we make use only of elementary facts of complex and Fourier analysis and basic tools which follow from the geometrical properties of the underlying Banach space. Moreover, our characterisations can be treated as a natural extension to UMD spaces of some known results for Hilbert spaces, see, e.g., [27, Theorem 4.1] and [19, Theorem 4.3]. In Sect. 4, we use these characterisations to provide a strikingly elementary proof of the group decomposition of cosine functions on UMD spaces, see Theorem 4.1.

The second part of the paper addresses the similarity problem for cosine function generators on Hilbert spaces. In Sect. 5, we prove a cosine function analogue of McIntosh's characterisation of the boundedness of the H^∞ functional calculus for sectorial operators on Hilbert spaces, see [41]. Our proof that the estimates of a square function type imply the boundedness of the H^∞ functional calculus on a horizontal parabola, i.e., the proof of $(ii) \Rightarrow (iii)$ in Theorem 5.1 differs from the original approach given by McIntosh. It follows, in principle, the line of the proof of the related sectorial result proposed by Kunstmann and Weis in [36, Theorem 11.9 $(H4) \Rightarrow (H1)$]. However, the details are more difficult and in particular the proof requires a deep result from the theory of Cauchy singular operators on Carleson curves. Moreover, we show that every generator of a cosine function on a Hilbert space X has a dilation to a multiplication operator on $L_2(\mathbb{R}; X)$, see Proposition 5.3.

Finally, the last part of the paper is devoted to the study of the exponential growth bound of cosine functions. Theorem 6.1 gives an affirmative answer to Fattorini's question on the growth bounds of perturbed cosine functions on Hilbert spaces, see [19, p. 240] and the beginning of Sect. 6 for more details. The positive answer seems to be contrary to his hypothesis, see the end of [19, Section 3]. We prove a cosine function analogue of the generation theorem for C_0-semigroups, which is independently due to Gomilko [25] and to Feng and Shi [21], and which is of independent interest, see Theorems 2.3 and 2.4 in Sect. 2. In the remainder part of the proof of Theorem 6.1, we make use of the Carleson embedding theorem for the vector-valued Hardy space $H_2(\mathbb{C}_+; X)$, which allows us also to give cosine function analogues of Datko's and Gearhart's theorem for C_0-semigroups, see Theorem 6.3.

We conclude with a few additional comments on the organisation of the paper. The characterisations mentioned above in the description of the first and third part of the paper are stated without proofs in Sect. 2. These results are closely related to each other and it seems to be more appropriate to collect them together. Then, for their proofs, we refer the reader to Sect. 3. Moreover, for the convenience of the reader, to place our results in a context, we also include in Sect. 2 some known facts, which we use in the sequel. Remarks and examples which complete our results and which are of independent interest are also included.

2. Widder's growth condition & d'Alembert's equation

Recall that the first well-known generation conditions in the theory of cosine functions and C_0-semigroups are based on the classical Widder theorem, which says that a function $r \in \mathcal{C}^{\infty}((\sigma, \infty); \mathbb{C})$ has a Laplace representation, i.e., $r = \mathcal{L}f$ on (σ, ∞) for some $f : \mathbb{R}_+ \to \mathbb{C}$ with $\operatorname{esssup}_{t \geq 0} |e^{-\sigma t} f(t)| < \infty$, if and only if r satisfies Widder's growth condition:

$$\|r\|_{W,\sigma} := \sup_{k \in \mathbb{N}_0, \lambda > \sigma} \frac{(\lambda - \sigma)^{k+1}}{k!} \left| r^{(k)}(\lambda) \right| < \infty. \tag{2.1}$$

The corresponding generation condition of Widder's type for cosine function generators was established independently by Da Prato and Giusti, Sova and Fattorini, see, e.g., [2, Theorem 3.15.3] for a precise formulation of this result.

Since it is usually impossible in applications to verify whether or not a function r satisfies Widder's growth condition (2.1), some complex conditions are discussed in the literature which are sufficient for a holomorphic function $r : \{\operatorname{Re} \lambda > \sigma\} \to \mathbb{C}$ to have the Laplace representation. For instance, there exists a natural connection between the theory of Hardy spaces and semigroup theory which was first pointed out by Hille and Phillips, see [34, Sections 6 and 12]. In particular, [34, Theorem 12.6.1] provides a sufficient condition for a closed linear operator to be the generator of an immediately continuous C_0-semigroup.

More recently, Gomilko [25] and Feng and Shi [21] independently applied a similar idea and obtained the following generation condition for C_0-semigroup generators.

THEOREM 2.1. [21,25] *Let A be a densely defined linear operator on a Banach space X. Suppose that there exists $\sigma \geq 0$ such that $\{\operatorname{Re} \lambda > \sigma\} \subset \rho(A)$ and*

$$\sup_{\xi > \sigma} (\xi - \sigma) \int_{\operatorname{Re} \lambda = \xi} \left| \left(x^*, R(\lambda, A)^2 x \right) \right| |\mathrm{d}\lambda| < \infty \tag{2.2}$$

for every $x \in X$ and $x^* \in X^*$. Then, A generates a C_0-semigroup T on X such that $\sup_{t \geq 0} \|e^{-\sigma t} T(t)\| < \infty$.

By an analysis of the proof of [34, Theorem 12.6.1], or [25, Theorem], one can abstract the following condition which is sufficient to ensure that an operator-valued function satisfies Widder's growth condition.

THEOREM 2.2. *Let X be a Banach space and let $\sigma \geq 0$. Let $h : \{\operatorname{Re} \lambda > \sigma\} \to \mathcal{L}(X)$ be a holomorphic function such that*

$$\sup_{\xi > \sigma} (\xi - \sigma) \int_{\operatorname{Re} \lambda = \xi} \left| \left(x^*, h'(\lambda) x \right) \right| |\mathrm{d}\lambda| < \infty \tag{2.3}$$

for every $x \in X$ and $x^* \in X^*$. Then,

$$\sup_{\lambda > \sigma, k \in \mathbb{N}} \frac{(\lambda - \sigma)^{k+1}}{k!} \left\| h^{(k)}(\lambda) \right\| < \infty. \tag{2.4}$$

If, additionally, $h(\lambda) \to 0$ as $\lambda \to \infty$ in the weak operator topology, then h satisfies Widder's growth condition, i.e., $\|h\|_{W,\sigma} < \infty$.

Combining Theorem 2.2 with the Fattorini-Da Prato-Giusti-Sova theorem, we easily get the following cosine function analogue of the Gomilko-Feng-Shi generation result for C_0-semigroups.

THEOREM 2.3. *Let A be a densely defined linear operator on a Banach space X and $\sigma \geq 0$. Assume that $\{\lambda^2 : \operatorname{Re}\lambda > \sigma\} \subset \rho(A)$ and*

$$\sup_{\xi>\sigma}(\xi - \sigma) \int_{\operatorname{Re}\lambda=\xi} \left| \left(x^*, R(\lambda^2, A)x - 2\lambda^2 R(\lambda^2, A)^2 x\right) \right| |\mathrm{d}\lambda| < \infty \tag{2.5}$$

for every $x \in X$ and $x^* \in X^*$. Then, A generates a cosine function C on X such that $\sup_{t\geq 0} \|e^{-\sigma t} C(t)\| < \infty$.

The condition (2.5) is also necessary for cosine function generators on Hilbert spaces (similarly as (2.2) in the case of C_0-semigroup generators) and by standard arguments based on Plancherel's theorem, this characterisation result can be stated in the following form.

THEOREM 2.4. *Let A be a densely defined linear operator on a Hilbert space X and $\sigma \geq 0$. Then, the following assertions are equivalent:*

(i) *A generates a cosine function C on X such that $\sup_{t\geq 0} \|e^{-\sigma t} C(t)\| < \infty$.*
(ii) *$\{\lambda^2 : \operatorname{Re}\lambda > \sigma\} \subset \rho(A)$ and for every $x \in X$*

$$\sup_{\xi>\sigma}(\xi - \sigma) \int_{\operatorname{Re}\lambda=\xi} \left\| \lambda R\big(\lambda^2, A\big)x \right\|^2 |\mathrm{d}\lambda| < \infty,$$
$$\sup_{\xi>\sigma}(\xi - \sigma) \int_{\operatorname{Re}\lambda=\xi} \left\| \lambda R\big(\lambda^2, A\big)^* x \right\|^2 |\mathrm{d}\lambda| < \infty. \tag{2.6}$$

Theorem 2.4 plays a crucial role in the proof of Theorem 6.1 in Sect. 6.

Recall that a Banach space X is called a UMD space if the Hilbert transform is a bounded operator on $L_p(\mathbb{R}; X)$, $1 < p < \infty$. Any space of the form $L_p(\Omega, \mu)$, $1 < p < \infty$, is UMD. We refer the reader to [44] or [1] for more details about UMD spaces and the Hilbert transform.

The following characterisations of generators of cosine functions and C_0-groups on UMD spaces are the main results of this section.

THEOREM 2.5. *Let A be a densely defined linear operator on a UMD space X. Then, the following assertions are equivalent:*

(i) *A generates a cosine function on X.*
(ii) *There exist $\sigma > 0$ and $m \in \mathbb{N}_0$ such that $\{\lambda^2 : \operatorname{Re}\lambda \geq \sigma\} \subset \rho(A)$, $\sup_{\operatorname{Re}\lambda\geq\sigma} \|\lambda^{-m} R(\lambda^2, A)\| < \infty$, and*

$$\int_{\sigma+i\mathbb{R}} e^{\lambda t} \left(x^*, \lambda R(\lambda^2, A)x\right) \mathrm{d}\lambda \tag{2.7}$$

exists in the improper sense for every $t > 0$, $x \in X$ and $x^* \in X^*$.

THEOREM 2.6. *Let B be a densely defined linear operator on a UMD space X. Then, the following assertions are equivalent:*

(i) *B generates a C_0-group on X.*

(ii) *There exist $\sigma > 0$ and $m \in \mathbb{N}_0$ such that $\{|\mathrm{Re}\,\lambda| \geq \sigma\} \subset \rho(B)$, $\sup_{|\mathrm{Re}\,\lambda|\geq\sigma} \|\lambda^{-m} R(\lambda, B)\| < \infty$, and*

$$\int_{\sigma+i\mathbb{R}} e^{\lambda t}\left(x^*, R(\lambda, \pm B)x\right) \mathrm{d}\lambda \tag{2.8}$$

exists in the improper sense for every $t > 0$, $x \in X$ and $x^* \in X^*$.

Before going into details of the proofs, we conclude this section with a few remarks on our generation conditions and their connections with related characterisations from the literature.

REMARKS 2.7. (a) In the comment after [25, Proposition], Gomilko noticed that the condition (2.2) for $\sigma = 0$ is not necessary even for a bounded operator A on a reflexive Banach space X which generates a uniformly bounded C_0-(semi)group, see also [4, Section 3]. More precisely, the example quoted there shows that there exist $x \in X$ and $x^* \in X^*$ such that $\sup_{\xi>0} \xi \|(x^*, R(\cdot, A)^2 x)\|_{L_1(\xi+i\mathbb{R};\mathbb{C})} = \infty$. Since, in this case, A is bounded, it is clear that $\sup_{\xi>\sigma} (\xi - \sigma)\|R(\cdot, A)^2\|_{L_1(\xi+i\mathbb{R};\mathcal{L}(X))} < \infty$ for all $\sigma > 0$. Of course the necessity of (2.2) is related to the geometrical properties of the underlying Banach space X. For instance, it is not difficult to give an example of a generator of a uniformly bounded C_0-(semi)group for which (2.2) does not hold for any $\sigma \geq 0$. Indeed, it is easy to check that every C_0-(semi)group on a Banach space X for which the representation by the *complex inversion formula* on X does not hold provides an example of a C_0-(semi)group generator A such that $(x^*, R(\cdot, A)^2 x)$ is not integrable on $\sigma + i\mathbb{R}$ for any $\sigma > \omega(T)$ and for some $x \in X$ and $x^* \in X^*$ even in the improper sense. As an example we can consider the shift group on $X = L_1(\mathbb{R})$, see e.g. [14, Proposition 6]. Recall also that the complex inversion formula for C_0-semigroups holds on every UMD space, see [14] or [2, Section 3.12]. In particular, it shows that there exists a generator A of a C_0-(semi)group such that for any $\sigma > 0$ and $p > 1$ the following condition

$$R(\cdot, A)x \in L_p(\sigma + i\mathbb{R}; X) \quad \text{and} \quad R(\cdot, A)^* x^* \in L_{p'}(\sigma + i\mathbb{R}; X^*)$$

does not hold for any $x \in X$ and $x^* \in X^*$, where $p' = \frac{p}{p-1}$.

(b) On the other hand, by Theorem 2.1 and a Baire category argument, one can show that for a densely defined linear operator A on a Hilbert space X with $\{\mathrm{Re}\,\lambda > \sigma\} \subset \rho(A)$ for some $\sigma \in \mathbb{R}$, the following condition for $\alpha = \frac{1}{2}$:

$$\begin{aligned} \|R(\cdot, A)x\|_{L_2(\xi+i\mathbb{R};X)} &= O(1/\xi^\alpha), \\ \|R(\cdot, A)^* x\|_{L_2(\xi+i\mathbb{R};X)} &= O(1/\xi^\alpha), \quad x \in X, \end{aligned} \tag{2.9}$$

as $\xi \to \infty$ is sufficient for A to be the generator of a C_0-semigroup on X. Clearly, by Plancherel's theorem, (2.9) is also necessary.

Adapting Krein's construction from [38], see also [36, Example 3.4], one can show that the rate of decay of the L_2 norms in (2.9) (with $\alpha = \frac{1}{2}$) cannot be weakened, in general. Indeed, let $X := L_2(0,\infty) \times L_2(0,\infty)$ be equipped with the norm given by $\|(f,g)\|^2 := \|f\|_{L_2}^2 + \|g\|_{L_2}^2$, $f, g \in L_2(0,\infty)$, and let $A_\beta := \begin{pmatrix} -M & -M^\beta \\ 0 & -M \end{pmatrix}$, where M is the multiplication operator on $L_2(0,\infty)$ with maximal domain $\mathcal{D}(M)$, given by $(Mf)(s) := sf(s)$ $(s > 0,\ f \in \mathcal{D}(M))$. Then it is straightforward to see that for every $\alpha < \frac{1}{2}$ there exists β such that (2.9) holds for $A := A_\beta$, however, A_β is not the generator of a C_0-semigroup on X. (See also [36, Theorem 3.3].)

(c) It is interesting that the situation is completely different in the case of the generation conditions for cosine functions and C_0-groups. Namely, in [19, Theorem 4.3] Fattorini stated without proof that a weaker asymptotic behaviour of the resolvent of a densely defined linear operator A on a Hilbert space X, weaker than the one from (ii) of Theorem 2.4, is sufficient to ensure that A generates a cosine function.

THEOREM 2.8. [19] *Let A be a densely defined linear operator on a Hilbert space X and* $\sigma > 0$. *Then, the following assertions are equivalent:*

(i) *A generates a cosine function C on X satisfying*

$$\int_0^\infty \|e^{-\sigma t} C(t)x\|^2 \mathrm{d}t < \infty, \qquad \int_0^\infty \|e^{-\sigma t} C(t)^* x\|^2 \mathrm{d}t < \infty$$

for every $x \in X$.

(ii) $\{\lambda^2 : \operatorname{Re}\lambda \geq \sigma\} \subset \rho(A)$, $\quad R((\cdot)^2, A)x,\ R((\cdot)^2, A)^*x \in H_2(\sigma; X)$ *and*

$$\int_{\operatorname{Re}\lambda=\sigma} \left\|\lambda R(\lambda^2, A)x\right\|^2 |\mathrm{d}\lambda| < \infty, \qquad \int_{\operatorname{Re}\lambda=\sigma} \left\|\lambda R(\lambda^2, A)^* x\right\|^2 |\mathrm{d}\lambda| < \infty \tag{2.10}$$

for every $x \in X$.

As far as the author knows, the proof of this result has not appeared in the literature. This result is disregarded in Fattorini's monograph [20] and in the recent systematic survey on cosine function theory, due to Vasil'ev and Piskarev [46], where other results from the paper [19] are included. Note that Theorem 2.8 is a special case of our characterisation of cosine function generators on UMD spaces, Theorem 2.5. We point out that this Fattorini result was the main motivation of our studies in the present paper.

3. Proofs of the results from Sect. 2

The proof of Theorem 2.2 follows immediately from the Cauchy integral representation of r'. We include the proof for the convenience of the reader.

Proof of Theorem 2.2. Fix $x \in X$ and $x^* \in X^*$. Let $r(\lambda) := (x^*, h(\lambda)x)$, $\operatorname{Re}\lambda > \sigma$ and set $M := \sup_{\xi>\sigma}(\xi-\sigma)\int_{\operatorname{Re}\lambda=\xi}\left|(x^*, h'(\lambda)x)\right||\mathrm{d}\lambda|$. Note that for every $\xi > \sigma$, the function $r'(\cdot+\sigma) : \{\operatorname{Re}\lambda > \xi-\sigma\} \to \mathbb{C}$ belongs to $H_1(\xi-\sigma;\mathbb{C})$. In particular, it is represented by the Cauchy integral, i.e.,

$$r'(\lambda+\sigma) = \frac{1}{2\pi i}\int_{\operatorname{Re}\mu=\xi-\sigma}\frac{r'(\mu+\sigma)}{\lambda-\mu}\mathrm{d}\mu, \qquad \operatorname{Re}\lambda > \xi-\sigma.$$

Thus,

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}r'(t+\sigma) = \frac{(-1)^k k!}{2\pi i}\int_{\operatorname{Re}\mu=\xi-\sigma}\frac{r'(\mu+\sigma)}{(t-\mu)^{k+1}}\mathrm{d}\mu$$

for every $k \geq 0$ and $t > \xi-\sigma$. Hence,

$$\begin{aligned}\frac{t^{k+2}}{(k+1)!}\left|\frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}}r(t+\sigma)\right| &\leq \frac{k!}{2\pi}\frac{t^{k+2}}{(k+1)!}\int_{\operatorname{Re}\mu=\xi-\sigma}\frac{|r'(\mu+\sigma)|}{|t-\mu|^{k+1}}|\mathrm{d}\mu|\\ &\leq \frac{1}{2\pi}\frac{t^{k+2}}{k+1}\frac{1}{(t-\xi+\sigma)^{k+1}}\int_{\operatorname{Re}\mu=\xi-\sigma}|r'(\mu+\sigma)||\mathrm{d}\mu|\\ &\leq \frac{1}{2\pi}\frac{t^{k+2}}{k+1}\frac{1}{(t-\xi+\sigma)^{k+1}}\frac{M}{\xi-\sigma}.\end{aligned}$$

Since for every $t > 0$ and $k \geq 0$, there exists $\xi > \sigma$ such that $t = (k+2)(\xi-\sigma)$, by the above estimate we get

$$\frac{t^{k+2}}{(k+1)!}\left|\frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}}r(t+\sigma)\right| \leq \frac{Me}{\pi}.$$

Suppose now that $r(\lambda) \to 0$ as $\lambda \to \infty$. Then, since $\sup_{t>0}|t^2\frac{\mathrm{d}}{\mathrm{d}t}r(t+\sigma)| \leq \frac{Me}{\pi}$,

$$|r(t+\sigma)| = \left|\int_t^\infty \frac{\mathrm{d}}{\mathrm{d}s}r(s+\sigma)\mathrm{d}s\right| \leq \frac{Me}{\pi}\frac{1}{t}, \qquad t > 0.$$

Therefore, by the Uniform Boundedness Principle, the proof is complete. □

Proof of Theorem 2.3. Let $h(\lambda) := \lambda R(\lambda^2, A)$, $\operatorname{Re}\lambda > \sigma$. Note that $h'(\lambda) = R(\lambda^2, A) - 2\lambda^2 R(\lambda^2, A)^2$. Hence, by the first part of Theorem 2.2, we get that (2.4) holds. Since $h(\lambda) = h(\sigma_0) + \int_{\sigma_0}^{\lambda} h'(t)\mathrm{d}t$ for $\lambda > \sigma_0 > \sigma$, (2.4) implies that the family $\{h(\lambda)\}_{\lambda>\sigma_0}$ is uniformly bounded. This allows one to show that $h(\lambda)x \to 0$ as $\lambda \to \infty$ for every $x \in X$. Applying again Theorem 2.2, we get that $\|h\|_{W,\sigma} < \infty$. Therefore, A satisfies the generation condition of the Fattorini-Da Prato-Giusti-Sova theorem, see [2, Theorem 3.15.3], and the proof is complete. □

Proof of Theorem 2.4. The implication $(i) \Rightarrow (ii)$ is an immediate consequence of Plancherel's theorem. For the converse, by the Cauchy–Schwarz inequality, (2.6) implies that

$$\sup_{\xi>\sigma}(\xi-\sigma)\int_{\operatorname{Re}\lambda=\xi}\left|(2\lambda^2 R(\lambda^2, A)^2 x, x^*)\right||\mathrm{d}\lambda| < \infty$$

for every $x, x^* \in X$. Moreover, note that

$$\int_{\operatorname{Re}\lambda=\xi} \left|\left(R(\lambda^2, A)x, x^*\right)\right| |\mathrm{d}\lambda| \\ \le \left(\int_{\operatorname{Re}\lambda=\xi} \|\lambda R(\lambda^2, A)x\|^2 |\mathrm{d}\lambda|\right)^{1/2} \left(\int_{\operatorname{Re}\lambda=\xi} \frac{1}{|\lambda|^2} |\mathrm{d}\lambda|\right)^{1/2} \|x^*\|.$$

Since $\int_{\operatorname{Re}\lambda=\xi} \frac{1}{|\lambda|^2}|\mathrm{d}\lambda| < \frac{\pi}{\xi-\sigma}$ for every $\xi > \sigma$, (2.5) of Theorem 2.3 holds. □

Now, we go to the proofs of our main characterisations of generators of cosine functions and C_0-groups on UMD spaces. The following lemma is crucial for our approach.

LEMMA 3.1. *Let X be a Banach space. Let $S : \mathbb{R} \to \mathcal{L}(X)$ satisfy one of the following functional equations:*

(i) *(d'Alembert's equation) $S(t+s) + S(t-s) = 2S(t)S(s)$, $t, s \in \mathbb{R}$,*
(ii) *(Cauchy's equation) $S(t+s) = S(t)S(s)$, $t, s \in \mathbb{R}$.*

Then, strong measurability of S implies its strong continuity.

For the proof of Lemma 3.1, we refer the reader to [16, Lemma 5.2] or [20, Theorem 1.1, p. 24] and [15, Section VIII.1.3].

Proof of Theorem 2.5. $(i) \Rightarrow (ii)$: For the existence of (2.7) see, e.g., [10,9,30], where the representation of cosine functions on UMD spaces by means of the complex inversion formula was established. The remaining conditions of (ii) follow simply from (1.1).
$(ii) \Rightarrow (i)$: Let $S : \mathbb{R} \to \mathcal{L}(X)$ be defined in the following way: $S(t) := C(|t|)$ for $t \neq 0$ and $S(0) := I$, where $C(t)$, $t > 0$, is given by

$$(x^*, C(t)x) := \lim_{a\to\infty} \frac{1}{2\pi i} \int_{\sigma-ia}^{\sigma+ia} e^{\lambda t}(x^*, \lambda R(\lambda^2, A)x)\mathrm{d}\lambda \quad (x \in X,\ x^* \in X^*).$$

Note that for every $n \in \mathbb{N}$, $k = 1, \ldots, n$, $\operatorname{Re}\lambda \ge \sigma$ and $x \in \mathcal{D}(A^n)$, we have

$$\lambda R(\lambda^2, A)x = \sum_{j=0}^{k-1} \frac{1}{\lambda^{2j+1}} A^j x + \frac{1}{\lambda^{2k-1}} R(\lambda^2, A)A^k x. \tag{3.1}$$

In particular, by our assumptions, (3.1) yields

$$\frac{1}{(\cdot)^{2k+1}} R((\cdot)^2, A)y \in L_1(\sigma' + i\mathbb{R}; X) \tag{3.2}$$

for every $y \in \mathcal{D}(A^{m+2-k})$, $k = 0, \ldots, m+1$ and $\sigma' \ge \sigma$. Applying Cauchy's theorem, it is straightforward to show that

$$C(t)x = \frac{1}{2\pi i} \int_{\sigma'+i\mathbb{R}} e^{\lambda t} \lambda R(\lambda^2, A)x\mathrm{d}\lambda = x + \frac{1}{2\pi i} \int_{\sigma'+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda} R(\lambda^2, A)Ax\mathrm{d}\lambda \\ = x + \frac{t^2}{2}Ax + \frac{1}{2\pi i} \int_{\sigma'+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda^3} R(\lambda^2, A)A^2x\mathrm{d}\lambda \tag{3.3}$$

for every $t > 0$, $\sigma' \geq \sigma$ and $x \in \mathcal{D}(A^{m+4})$. Note that the first integral in (3.3) exists as an improper integral in X; the other ones are absolutely convergent.

Therefore, since $\mathcal{D}(A^{m+4})$ is dense in X and the operators $\frac{1}{2\pi i}\int_{\sigma-ia}^{\sigma+ia} e^{\lambda t}\lambda R(\lambda^2, A)\,\mathrm{d}\lambda$, $a > 0$, are uniformly bounded, they actually strongly converge to $C(t)$ as $a \to \infty$ for every $t > 0$. In particular, this shows that S is strongly measurable, and since A is closed, $S(t)$ and A commute for every $t \in \mathbb{R}$.

Now, we show that the function S satisfies d'Alembert's equation, i.e.,

$$S(t+s)x + S(t-s)x = 2S(t)S(s)x \tag{3.4}$$

for every $x \in X$ and for every $t, s \in \mathbb{R}$. Then, by Lemma 3.1, S is a cosine function on X, and by the uniqueness theorem for the Fourier transform, one can easily show that A is its generator.

First, note that by the definition of S, it suffices to verify (3.4) only for $t \geq s > 0$, if we additionally prove that the operators $S(t)$ and $S(s)$ commute for $t, s > 0$. For, by (3.2) and (3.3), we can write:

$$\begin{aligned} C(t)C(s)x &= C(s)x + C(t)x - x \\ &\quad + \frac{1}{(2\pi i)^2}\int_{\sigma+i\mathbb{R}}\int_{\sigma'+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda}\frac{e^{\mu s}}{\mu} R(\lambda^2, A)R(\mu^2, A)A^2 x\,\mathrm{d}\mu\mathrm{d}\lambda \end{aligned}$$

for every $t, s > 0$, $\sigma' \geq \sigma$ and $x \in \mathcal{D}(A^{m+4})$. Since $\mathcal{D}(A^{m+4})$ is dense in X, by Fubini's theorem, $C(t)$ and $C(s)$ commute.

Therefore, fix $t \geq s > 0$, $\sigma' > \sigma$ and $x \in \mathcal{D}(A^{m+4})$. Set

$$I := \frac{2}{(2\pi i)^2}\int_{\sigma+i\mathbb{R}}\int_{\sigma'+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda}\frac{e^{\mu s}}{\mu} R(\lambda^2, A)R(\mu^2, A)A^2 x\,\mathrm{d}\mu\mathrm{d}\lambda.$$

Note that the resolvent equation yields

$$\begin{aligned} 2\lambda\mu R(\lambda^2, A)R(\mu^2, A) &= \frac{1}{\mu-\lambda}\lambda R(\lambda^2, A) + \frac{1}{\lambda-\mu}\mu R(\mu^2, A) \\ &\quad + \frac{1}{\lambda+\mu}\mu R(\mu^2, A) + \frac{1}{\lambda+\mu}\lambda R(\lambda^2, A) \end{aligned}$$

for every $\lambda \neq \mu$, $\operatorname{Re}\lambda, \operatorname{Re}\mu \geq \sigma$. Hence, $I = I_1 + I_2 + I_3 + I_4$, where:

$$\begin{aligned} I_1 &:= \frac{1}{(2\pi i)^2}\int_{\sigma+i\mathbb{R}}\int_{\sigma'+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda^2}\frac{e^{\mu s}}{\mu^2}\frac{1}{\mu-\lambda}\lambda R(\lambda^2, A)A^2 x\,\mathrm{d}\mu\mathrm{d}\lambda, \\ I_2 &:= \frac{1}{(2\pi i)^2}\int_{\sigma+i\mathbb{R}}\int_{\sigma'+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda^2}\frac{e^{\mu s}}{\mu^2}\frac{1}{\lambda-\mu}\mu R(\mu^2, A)A^2 x\,\mathrm{d}\mu\mathrm{d}\lambda, \\ I_3 &:= \frac{1}{(2\pi i)^2}\int_{\sigma+i\mathbb{R}}\int_{\sigma'+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda^2}\frac{e^{\mu s}}{\mu^2}\frac{1}{\lambda+\mu}\mu R(\mu^2, A)A^2 x\,\mathrm{d}\mu\mathrm{d}\lambda, \\ I_4 &:= \frac{1}{(2\pi i)^2}\int_{\sigma+i\mathbb{R}}\int_{\sigma'+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda^2}\frac{e^{\mu s}}{\mu^2}\frac{1}{\mu+\lambda}\lambda R(\lambda^2, A)A^2 x\,\mathrm{d}\mu\mathrm{d}\lambda. \end{aligned}$$

For I_1, note that $\frac{1}{2\pi i}\int_{\sigma'+i\mathbb{R}} \frac{e^{\mu s}}{\mu^2}\frac{1}{\mu-\lambda}\mathrm{d}\mu = \frac{e^{\lambda s}}{\lambda^2} - \frac{s\lambda+1}{\lambda^2}$. Therefore, (3.3) gives

$$
\begin{aligned}
I_1 &= C(t+s)x - x - \frac{1}{2}(t+s)^2Ax - \frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda^3}R(\lambda^2, A)A^2x\mathrm{d}\lambda \\
&\quad - \frac{s}{2\pi i}\int_{\sigma+i\mathbb{R}} frace^{\lambda t}\lambda^2 R(\lambda^2, A)A^2x\mathrm{d}\lambda \\
&= C(t+s)x - C(t)x - tsAx - \frac{1}{2}s^2Ax - \frac{s}{2\pi i}\int_{\sigma+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda^2}R(\lambda^2, A)A^2x\mathrm{d}\lambda.
\end{aligned}
$$

For I_2, I_3 and I_4, we have that $\frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda^2}\frac{1}{\lambda-\mu}\mathrm{d}\lambda = -\frac{t\mu+1}{\mu^2}$, $\frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda^2}\frac{1}{\lambda+\mu}\mathrm{d}\lambda = \frac{e^{-\mu t}}{\mu^2} + \frac{t\mu-1}{\mu^2}$ and $\frac{1}{2\pi i}\int_{\sigma'+i\mathbb{R}} \frac{e^{\mu s}}{\mu^2}\frac{1}{\mu+\lambda}\mathrm{d}\mu = \frac{e^{-\lambda s}}{\lambda^2} + \frac{s\lambda-1}{\lambda^2}$, respectively. Applying (3.3), and in the case $t = s$, additionally, Cauchy's theorem and our assumptions on the resolvent of A, we get

$$
\begin{aligned}
I_2 &= -C(s)x + x + \frac{1}{2}s^2Ax - \frac{t}{2\pi i}\int_{\sigma'+i\mathbb{R}} \frac{e^{\mu s}}{\mu^2}R(\mu^2, A)A^2x\mathrm{d}\mu, \\
I_3 &= -C(s) + x + \frac{1}{2}s^2Ax + \frac{t}{2\pi i}\int_{\sigma'+i\mathbb{R}} \frac{e^{\mu s}}{\mu^2}R(\mu^2, A)A^2x\mathrm{d}\mu, \\
I_4 &= C(t-s)x - x - \frac{1}{2}(t-s)^2Ax - C(t)x + x + \frac{1}{2}t^2Ax \\
&\quad + \frac{s}{2\pi i}\int_{\sigma+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda^2}R(\lambda^2, A)A^2x\mathrm{d}\lambda.
\end{aligned}
$$

Combining now the above expressions for the integrals I_1, I_2, I_3, I_4, it is straightforward to see that (3.4) holds for every $t \geq s > 0$ and $x \in \mathcal{D}(A^{m+4})$. Since $\mathcal{D}(A^{m+4})$ is dense in X, this equation holds for all $x \in X$. Thus, the proof of $(ii) \Rightarrow (i)$ is complete. □

The proof of Theorem 2.6 is based on the same idea as the proof of Theorem 2.5; however, it is slightly more technical. For the convenience of the reader, we include some details.

Proof of Theorem 2.6. For the necessity of (2.8), we refer to [14], see also [30] or [10]. The necessity of the remaining conditions is clear. For the sufficiency of (ii), let $S : \mathbb{R} \to \mathcal{L}(X)$ be defined by $S(t) = \begin{cases} U(t), & t > 0, \\ I, & t = 0, \\ V(-t), & t < 0, \end{cases}$ where for every $t > 0$, the operators $U(t)$ and $V(t)$ are given by

$$
\begin{aligned}
(x^*, U(t)x) &:= \lim_{a\to\infty} \frac{1}{2\pi i}\int_{\sigma-ia}^{\sigma+ia} e^{\lambda t}(x^*, R(\lambda, B)x)\,\mathrm{d}\lambda, \\
(x^*, V(t)x) &:= \lim_{a\to\infty} \frac{1}{2\pi i}\int_{\sigma-ia}^{\sigma+ia} e^{\lambda t}(x^*, R(\lambda, -B)x)\,\mathrm{d}\lambda \quad (x \in X,\ x^* \in X^*).
\end{aligned}
$$

Note that for every $n \in \mathbb{N}$, $k = 1, \dots, n$, $|\mathrm{Re}\,\lambda| \geq \sigma$ and $x \in \mathcal{D}(B^n)$

$$R(\lambda, \pm B)x = \sum_{j=0}^{k-1} \frac{1}{\lambda^{j+1}} (\pm B)^j x + \frac{1}{\lambda^k} R(\lambda, \pm B)(\pm B)^k x. \tag{3.5}$$

In particular, our assumptions show that

$$\frac{1}{(\cdot)^k} R(\cdot, \pm B) y \in L_1(\sigma' + i\mathbb{R}; X) \tag{3.6}$$

for every $n \geq 2$, $k = 1, \dots, m+n$, $\sigma' \geq \sigma$ and $y \in \mathcal{D}(B^{m+n-k})$.

Combining now (3.5) and (3.6), for every $k = 1, \dots, 4$, $\sigma' \geq \sigma$, and $x \in \mathcal{D}(B^{m+4})$, we get

$$S(u)x = \sum_{j=0}^{k-1} \frac{u^j}{j!} B^j x + \frac{1}{2\pi i} \int_{\sigma' + i\mathbb{R}} \frac{e^{u\lambda}}{\lambda^k} R(\lambda, B) B^k x \,\mathrm{d}\lambda, \quad u > 0, \tag{3.7}$$

and

$$S(u)x = \sum_{j=0}^{k-1} \frac{u^j}{j!} B^j x + \frac{(-1)^{k+1}}{2\pi i} \int_{\sigma' + i\mathbb{R}} \frac{e^{-u\lambda}}{\lambda^k} R(-\lambda, B) B^k x \,\mathrm{d}\lambda, \quad u < 0. \tag{3.8}$$

By similar arguments as in the proof of Theorem 2.5, it remains to prove that S satisfies Cauchy's equation on $\mathbb{R}$, i.e.,

$$S(t+s)x = S(t)S(s)x \tag{3.9}$$

for every $x \in X$ and $s, t \in \mathbb{R}$. We prove (3.9) only for $s < 0 < t$. In the same manner, one can treat the other cases.

Fix $s < 0 < t$, $\sigma < \sigma'$ and $x \in \mathcal{D}(B^{m+4})$. Then, by (3.7), (3.8) and the resolvent equation, we obtain that

$$\begin{aligned} S(t)S(s)x &= S(s)x + tBS(s)x + \frac{1}{2\pi i}(I + sB) \int_{\sigma + i\mathbb{R}} \frac{e^{t\lambda}}{\lambda^2} R(\lambda, B) B^2 x \,\mathrm{d}\lambda \\ &\quad - \frac{1}{(2\pi i)^2} \int_{\sigma + i\mathbb{R}} \int_{\sigma' + i\mathbb{R}} \frac{e^{t\lambda}}{\lambda^2} \frac{e^{-s\mu}}{\mu^2} R(\lambda, B) R(-\mu, B) B^4 x \,\mathrm{d}\mu \mathrm{d}\lambda \\ &= S(s)x + tBS(s)x + (I + sB)\left(S(t)x - x - tBx\right) \\ &\quad + \frac{1}{(2\pi i)^2} \int_{\sigma + i\mathbb{R}} \int_{\sigma' + i\mathbb{R}} \frac{e^{\lambda t}}{\lambda^2} \frac{e^{-\mu s}}{\mu^2} \frac{1}{\lambda + \mu} R(\lambda, B) B^4 x \,\mathrm{d}\mu \mathrm{d}\lambda \\ &\quad - \frac{1}{(2\pi i)^2} \int_{\sigma + i\mathbb{R}} \int_{\sigma' + i\mathbb{R}} \frac{e^{\lambda t}}{\lambda^2} \frac{e^{-\mu s}}{\mu^2} \frac{1}{\mu + \lambda} R(-\mu, B) B^4 x \,\mathrm{d}\mu \mathrm{d}\lambda \\ &=: S(s)x + tBS(s)x + (I + sB)\left(S(t)x - x - tBx\right) + I_1 - I_2. \end{aligned}$$

For I_1, note that $\frac{1}{2\pi i}\int_{\sigma'+i\mathbb{R}}\frac{e^{-\mu s}}{\mu^2}\frac{1}{\lambda+\mu}\mathrm{d}\mu = -\frac{1}{\lambda^2}-\frac{s}{\lambda}+\frac{e^{s\lambda}}{\lambda^2}$. Hence, by (3.6) and (3.7),

$$\begin{aligned}
I_1 &= -\frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}}\frac{e^{t\lambda}}{\lambda^4}R(\lambda,B)B^4x\mathrm{d}\lambda - sB\left(\frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}}\frac{e^{t\lambda}}{\lambda^3}R(\lambda,B)B^3x\mathrm{d}\lambda\right)\\
&\quad+\frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}}\frac{e^{(t+s)\lambda}}{\lambda^4}R(\lambda,B)B^4x\mathrm{d}\lambda\\
&=: -S(t)x+x+tBx+\frac{t^2}{2}B^2x+\frac{t^3}{6}B^3x-sBS(t)x+sBx\\
&\qquad+stB^2x+\frac{st^2}{2}B^3x+I_{1,0}.
\end{aligned}$$

By Cauchy's theorem and our assumptions on the resolvent of A, it is easily seen that $I_{1,0}=0$ in the case $t+s\le 0$, and

$$I_{1,0}=S(t+s)x-x-(t+s)Bx-\frac{(t+s)^2}{2}B^2x-\frac{(t+s)^3}{6}B^3x$$

if $t+s>0$. Finally, for I_2, note that $\frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}}\frac{e^{\lambda s}}{\lambda^2}\frac{1}{\lambda+\mu}\mathrm{d}\lambda = -\frac{1}{\mu^2}+\frac{t}{\mu}+\frac{e^{-t\mu}}{\mu^2}$. Hence,

$$\begin{aligned}
I_2 &= -\frac{1}{2\pi i}\int_{\sigma'+i\mathbb{R}}\frac{e^{-s\mu}}{\mu^4}R(-\mu,B)B^4x\mathrm{d}\lambda + tB\left(\frac{1}{2\pi i}\int_{\sigma'+i\mathbb{R}}\frac{e^{-s\mu}}{\mu^3}R(-\mu,B)B^3x\mathrm{d}\mu\right)\\
&\quad+\frac{1}{2\pi i}\int_{\sigma'+i\mathbb{R}}\frac{e^{-(t+s)\mu}}{\mu^4}R(-\mu,B)B^4x\mathrm{d}\mu\\
&=: S(s)x-x-(s+t)Bx-\left(\frac{s^2}{2}+ts\right)B^2x-\left(\frac{s^3}{6}+\frac{ts^2}{2}\right)B^3x+tBS(s)x+I_{2,0}.
\end{aligned}$$

Analogously, $I_{2,0}=0$ if $t+s\ge 0$, and

$$I_{2,0}=-S(t+s)+x+(t+s)Bx+\frac{(t+s)^2}{2}B^2x+\frac{(t+s)^3}{6}B^3x,$$

if $t+s<0$. Applying the above-obtained expressions for I_1 and I_2, it is easy to check that (3.9) holds for $s<0<t$ and $x\in\mathcal{D}(B^{m+4})$, and thus for every $x\in X$, by the density of $\mathcal{D}(B^{m+4})$. □

We conclude this section with a few remarks on the generation condition (ii) of Theorem 2.5. Analogous remarks can be stated also in the group case.

REMARKS 3.2. (a) Recall that every UMD space has nontrivial Fourier type. Therefore, if A is the generator of a cosine function C on a UMD space X with Fourier type p, then $(\cdot)R((\cdot)^2,A)x\in L_{p'}(\sigma+i\mathbb{R};X)$ for every $x\in X$ and $\sigma>\omega(C)$, where $p':=\frac{p}{p-1}$. It is of interest whether (2.7) in Theorem 2.5(ii) can be replaced, e.g., by an integral condition on the resolvent of the operator A and/or its adjoint, which is necessary for cosine function generators on UMD spaces. In particular, one can ask if the condition (ii) of Theorem 2.5 is equivalent to the following one:

(iii) *There exists* $\sigma > 0$ *such that* $\{\lambda^2 : \operatorname{Re}\lambda \geq \sigma\} \subset \rho(A)$, $\sup_{\operatorname{Re}\lambda \geq \sigma} \|R(\lambda^2, A)\| < \infty$, *and*

$$\int_{\operatorname{Re}\lambda=\sigma} \left\|\lambda R(\lambda^2, A)x\right\|^{p'} |\mathrm{d}\lambda| < \infty,$$
$$\int_{\operatorname{Re}\lambda=\sigma} \left\|\lambda R(\lambda^2, A)^* x^*\right\|^{p'} |\mathrm{d}\lambda| < \infty,$$

for every $x \in X$ *and* $x^* \in X^*$.

Clearly, by integration by parts, it holds if X is a Hilbert space.

It is also of interest to verify the q-integrability of $(\cdot)R((\cdot)^2, A)x$ on the vertical lines for $q = r, r'$, where $r > 1$, $1/r + 1/r' = 1$ and for A belonging to some class of cosine function generators on a UMD space, for instance, to the generators of positive cosine functions on L_p spaces, $p > 1$. However, it is not difficult to check that if B is the generator of a C_0-group U on a UMD space X with Fourier type p, and if *the strip type* of B,

$$\omega_{st}(B) := \inf\left\{\omega \geq 0 : \sigma(B) \subset \{|\operatorname{Re}\lambda| < \omega\}, \sup_{|\operatorname{Re}\lambda| \geq \omega} \|R(\lambda, B)\| < \infty\right\},$$

is strictly less than *the group type* of U,

$$\theta(U) := \inf\left\{\theta \geq 0 : \sup_{t\in\mathbb{R}} \|e^{-\theta|t|}U(t)\| < \infty\right\},$$

then for every $\sigma > \theta(U)$ there exists $x_\sigma \in X$ such that $R(\cdot, B)x_\sigma \notin L_p(\pm\sigma + i\mathbb{R}; X)$. Indeed, if $R(\cdot, B)x \in L_p(\sigma + i\mathbb{R}; X)$ for some $\sigma > \theta(U)$ and every $x \in X$, then by Cauchy's theorem and the complex inversion formula we get $\frac{1}{2\pi}(\mathcal{F}R(\omega - i\cdot, B)x)(t) = e^{-\omega t}U(t)x$, $t > 0$, for every $x \in X$ and $\omega_{st}(B) < \omega < \theta(U)$. Clearly, this contradicts Datko's theorem, see, e.g. [2, Theorem 5.1.2]. Moreover, the method of the proof of Theorem 4.1 below gives $(\cdot)R((\cdot)^2, B^2)x_\sigma \notin L_p(\sigma + i\mathbb{R}; X)$. Recall that B^2 generates a cosine function on X, see e.g. [2, Example 3.14.15]. For an example of a generator B of a C_0-group U with $\omega_{st}(B) < \theta(U)$, see the modification of Wolff's example due to Haase in [26, Section 5].

(b) Recall that if a function F is the Fourier transform of $f \in L_p(\mathbb{R}; \mathbb{C})$, $1 < p < 2$, then by a reciprocal formula due to Hille and Tamarkin, f is the Fourier transform of $\frac{1}{2\pi}(F)(-\cdot)$, i.e.,

$$f(t) = L_p - \lim_{a\to\infty} \frac{1}{2\pi} \int_{-a}^{a} e^{ist}F(s)\mathrm{d}s =: L_p - \lim_{a\to\infty} \frac{1}{2\pi} f(t, a), \tag{3.10}$$

see [35, Theorem, p. 772] or [33], [45, Theorem 108]. It is easily seen that the proof of the reciprocal formula (3.10) extends to the vector-valued case, i.e., when $f \in L_p(\mathbb{R}; X)$, $1 < p < 2$, where X is a UMD space with Fourier type

p. Moreover, for $p = 2$, the well-known Carleson theorem shows that $f(\cdot, a)$ converges to f almost everywhere on $\mathbb{R}$ as $a \to \infty$.

However, note that we cannot replace (2.7) in (ii) of Theorem 2.5 by the statement that for every $x \in X$ and $x^* \in X^*$, the function

$$F(\cdot) := (x^*, (\sigma + i\cdot)R((\sigma + i\cdot)^2, A)x)$$

is the Fourier transform of a function f in $L_p(\mathbb{R}; \mathbb{C})$, $1 < p \leq 2$, even if we additionally assume that for every $x \in X$ and $x^* \in X^*$, the partial Fourier transforms $f(\cdot, a)$, $a > 0$, converge almost everywhere on $\mathbb{R}$.

Indeed, note that the condition (2.7) cannot be omitted even in the case of a Hilbert space X. Roughly speaking, it is sufficient to consider the second derivative $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ restricted to some appropriate subspace X of $L_2(\mathbb{R})$ which is not non-invariant with respect to translations. For such an example of X, we refer the reader to [19, p. 251]; see also [36, Example 3.2]. This sheds some light on the role of the assumptions on the resolvent of the adjoint of A in Theorem 2.8 and [27, Theorem 4.1].

4. Square root reduction of cosine function generators

Now, we apply our characterisation of C_0-group generators, Theorem 2.6, to provide an alternative and elementary proof of the well-known Fattorini theorem on square root reduction for generators of cosine functions on UMD spaces.

THEOREM 4.1. [17] *Let A be the generator of a cosine function C on a UMD space X. Assume that $(\omega, \infty) \subset \rho(A)$ and $\sup_{\lambda>0} \|\lambda R(\lambda, A - \omega)\| < \infty$ for some $\omega \in \mathbb{R}$. Then, the operator $\mathcal{B} := i(\omega - A)^{1/2}$ generates a C_0-group on X and $\mathcal{B}^2 = A - \omega$.*

By the following lemmas, the proof of Theorem 4.1 follows in a straightforward way from well-established facts of the theory of sectorial operators and Theorem 2.6. The first lemma gives the well-known relationship between the resolvent of a linear operator and the resolvent of its square. The proof of it is obvious.

LEMMA 4.2. *Let $\mathcal{B}$ be a closed linear operator on a Banach space X. For $\lambda \in \mathbb{C}$, if $\lambda^2 \in \rho(\mathcal{B}^2)$, then $\pm\lambda \in \rho(\mathcal{B})$ and*

$$R(\lambda, \pm\mathcal{B}) = \lambda R(\lambda^2, \mathcal{B}^2) \pm \mathcal{B}R(\lambda^2, \mathcal{B}^2).$$

In particular,

$$R(\lambda, \mathcal{B}) = 2\lambda R(\lambda^2, \mathcal{B}^2) + R(-\lambda, \mathcal{B}).$$

The next lemma is a special version of the complex inversion theorem for cosine functions on UMD spaces.

LEMMA 4.3. *Let A be the generator of a cosine function C on a UMD space X and let $\sigma > \omega(C)$. Then,*

$$\lim_{a\to\infty}\int_{\sigma+i0}^{\sigma+ia} e^{\lambda t}\lambda R(\lambda^2, A)x \, d\lambda \quad \text{and} \quad \lim_{a\to\infty}\int_{\sigma-ia}^{\sigma+i0} e^{\lambda t}\lambda R(\lambda^2, A)x \, d\lambda$$

exist for every $x \in X$ and $t > 0$.

The proof of Lemma 4.3 follows the lines of the proof of the complex inversion formula for C_0-semigroups on UMD spaces, see [14, Theorem 1] and also [2, Theorem 3.12.2]. We leave the details for the reader.

For a recent account of the theory of sectorial operators, we refer the reader to [28].

Proof of Theorem 4.1. Since $A - \omega$ is also the generator of a cosine function on X, see, e.g., [2, Corollary 3.14.10], without loss of generality, we assume that $\omega = 0$. Fix $\sigma > \omega(C)$. By [28, Proposition 3.1.2], $(-A)^{1/2}$ is sectorial with angle less than $\frac{\pi}{2}$. In particular, there exists $M > 0$ such that

$$\|R(\lambda, (-A)^{\frac{1}{2}})\| \leq \frac{M}{|\lambda|}, \quad \operatorname{Re}\lambda < 0. \tag{4.1}$$

Let $\mathcal{B} := i(-A)^{1/2}$. By Lemma 4.2, $\{|\operatorname{Re}\lambda| \geq \sigma\} \subset \rho(\mathcal{B})$ and

$$R(\lambda, \mathcal{B}) = \lambda R(\lambda^2, A) + \mathcal{B}R(\lambda^2, A), \tag{4.2}$$

$$R(\lambda, \mathcal{B}) = 2\lambda R(\lambda^2, A) + R(-\lambda, \mathcal{B}), \qquad |\operatorname{Re}\lambda| \geq \sigma. \tag{4.3}$$

Moreover, by the moment inequality, see [28, Proposition 6.6.4], we get

$$\begin{aligned}\|\mathcal{B}R(\lambda^2, A)x\|^2 &\leq c\|R(\lambda^2, A)x\|\|AR(\lambda^2, A)x\| \\ &\leq c\|R(\lambda^2, A)x\|\|x\| + c\|\lambda R(\lambda^2, A)\|^2\end{aligned}$$

for every $x \in X$ and $|\operatorname{Re}\lambda| \geq \sigma$, where c is a suitable constant. Recall that $\sup_{\operatorname{Re}\lambda\geq\sigma}\|\lambda R(\lambda^2, A)\| < \infty$. Therefore, combining the above estimate with (4.2), we get $\sup_{|\operatorname{Re}\lambda|\geq\sigma}\|R(\lambda, \mathcal{B})\| < \infty$. Fix $x \in X$, $x^* \in X^*$ and $t > 0$. Note that

$$\begin{aligned}\int_{\sigma-ia}^{\sigma+i0} e^{\lambda t}(x^*, R(\lambda, \mathcal{B})x)d\lambda &= i\int_0^a e^{(\sigma-is)t}(x^*, R(\sigma - is, \mathcal{B})x)ds \\ &= \frac{i}{t}\int_0^a e^{(\sigma-is)t}(x^*, R(\sigma - is, \mathcal{B})^2x)ds \\ &\quad - \frac{e^{(\sigma-ia)t}}{t}(x^*, R(\sigma - ia, \mathcal{B})x) + \frac{e^{\sigma t}}{t}(x^*, R(\sigma, \mathcal{B})x).\end{aligned}$$

Since, by (4.1), $\|R(\lambda, \mathcal{B})\| \leq \frac{M}{|\lambda|}$, $\operatorname{Im}\lambda < 0$, it follows that

$$\lim_{a\to\infty}\int_{\sigma-ia}^{\sigma+i0} e^{\lambda t}(x^*, R(\lambda, \mathcal{B})x)d\lambda \quad \text{and} \quad \lim_{a\to\infty}\int_{\sigma+i0}^{\sigma+ia} e^{\lambda t}(x^*, R(-\lambda, \mathcal{B})x)d\lambda \tag{4.4}$$

exist. Combining Lemma 4.3, (4.4) and (4.3), we easily get the existence of

$$\lim_{a\to\infty}\int_{\sigma-ia}^{\sigma+ia} e^{\lambda t}(x^*, R(\lambda, \mathcal{B})x)d\lambda.$$

In the same manner, one can prove that $\lim_{a\to\infty}\int_{\sigma-ia}^{\sigma+ia} e^{\lambda t}(x^*, R(\lambda, -\mathcal{B})x)d\lambda$ exists. Hence, by Theorem 2.6, $\mathcal{B}$ generates a C_0-group on X and the proof is complete. □

5. McIntosh type characterisation of cosine function generators

In this section, we provide further characterisations of cosine function generators on Hilbert spaces, which correspond to the well-known results on sectorial operators with bounded H^∞ functional calculus.

We start with a cosine function analogue of McIntosh's characterisation of the boundedness of the H^∞ functional calculus for sectorial operators on Hilbert spaces, see [41]. For the background on the functional calculus of linear operators, we refer the reader to [28].

Set $\Pi_\omega := \{z \in \mathbb{C} : (\operatorname{Im} z)^2 < 4\omega^2 \operatorname{Re} z\}$ for $\omega > 0$.

THEOREM 5.1. *Let A be a densely defined linear operator on a Hilbert space X. Then, the following assertions are equivalent:*

(i) *A generates a cosine function C on X.*

(i′) *There exists $\sigma > 0$ such that $\sigma(\mathcal{A}) \subset \Pi_\sigma$, $\sup_{z\in\mathbb{C}\setminus\Pi_\sigma} \|R(z,\mathcal{A})\| < \infty$ and*

$$\int_{\partial\Pi_\sigma} \left\| z^{1/2} R(z,\mathcal{A})x \right\|^2 \frac{|\mathrm{d}z|}{|z|^{1/2}} < \infty, \quad \int_{\partial\Pi_\sigma} \left\| \left(z^{1/2} R(z,\mathcal{A})\right)^* x \right\|^2 \frac{|\mathrm{d}z|}{|z|^{1/2}} < \infty$$

for every $x \in X$, where $\mathcal{A} := \sigma^2 - A$.

(i″) *There exists $\sigma > 0$ such that $\mathcal{A} := \sigma^2 - A$ is sectorial, $\sigma(\mathcal{A}) \subset \Pi_\sigma$, $\sup_{z\in\mathbb{C}\setminus\Pi_\sigma} \|R(z,\mathcal{A})\| < \infty$ and for every $x \in X$*

$$\int_{\partial\Pi_\sigma} \left\| \mathcal{A}^{1/2} R(z,\mathcal{A})x \right\|^2 \frac{|\mathrm{d}z|}{|z|^{1/2}} < \infty, \quad \int_{\partial\Pi_\sigma} \left\| \left(\mathcal{A}^{1/2} R(z,\mathcal{A})\right)^* x \right\|^2 \frac{|\mathrm{d}z|}{|z|^{1/2}} < \infty.$$

(ii) *There exists $c, \sigma > 0$ such that $\mathcal{A} := \sigma^2 - A$ is sectorial, $\sigma(\mathcal{A}) \subset \Pi_\sigma$, $\sup_{z\in\mathbb{C}\setminus\Pi_\sigma} \|R(z,\mathcal{A})\| < \infty$ and*

$$\frac{1}{c}\|x\|^2 \le \int_{\partial\Pi_\sigma} \left\| \left[\mathcal{A}^{1/2} \pm z^{1/2}\right] R(z,\mathcal{A})x \right\|^2 \frac{|\mathrm{d}z|}{|z|^{1/2}} \le c\|x\|^2, \quad x \in X. \tag{5.1}$$

(iii) *There exists $\sigma > 0$ such that $\sigma(\mathcal{A}) \subset \Pi_\sigma$ and $\mathcal{A}$ has the bounded natural $H^\infty(\Pi_\sigma)$ functional calculus, where $\mathcal{A} := \sigma^2 - A$.*

The equivalence of (i) and (iii) is well-known see, e.g., [28, Theorem 7.4.7 and Corollary 7.4.6] and [32, Section 5] for related matters in the case of arbitrary UMD spaces.

The proof of the main implication $(ii) \Rightarrow (iii)$ of Theorem 5.1 is based on the theory of Cauchy transforms on Carleson curves. For the convenience of the reader, we quote here one of the main results of this theory, which will be basic for our further considerations. For its proof and also for the background on Carleson curves and Muckenhoupt weights used below, we refer the reader to [6, Sections 4 and 5] and [6, Sections 1 and 2], respectively.

Let Γ be a simple locally rectifiable curve. Set $\Gamma(u,\epsilon) := \Gamma \cap \{|z-u| \le \epsilon\}$ and $\Gamma_{u,\epsilon} := \Gamma \setminus \Gamma(u,\epsilon)$ for every $u \in \Gamma$ and $\epsilon > 0$. Let $\mathcal{C}_0^\infty(\Gamma)$ stands for the set of the restrictions of functions in $\mathcal{C}_0^\infty(\mathbb{R}^2)$ to Γ. For $\epsilon > 0$, $u \in \Gamma$ and $g \in \mathcal{C}_0^\infty(\Gamma)$, let $S_{\Gamma,\epsilon} g(u)$ denotes a truncated singular integral of g at the point u, i.e.,

$$S_{\Gamma,\epsilon} g(u) := \frac{1}{2\pi i} \int_{\Gamma_{u,\epsilon}} \frac{g(z)}{z-u} \mathrm{d}z.$$

Recall that for every $g \in \mathcal{C}_0^\infty(\Gamma)$, the limit

$$S_\Gamma g(u) := \lim_{\epsilon \to 0^+} S_{\Gamma,\epsilon} g(u) =: \frac{1}{2\pi i} \int_\Gamma \frac{g(z)}{z-u} \mathrm{d}z$$

exists for a.e. $u \in \Gamma$ and is called the value of the Cauchy singular integral of g at u.

LEMMA 5.2. *Let Γ be a Carleson curve and let w be a Muckenhoupt A_p-weight on Γ, i.e., $w \in A_p(\Gamma)$, $1 < p < \infty$. Then, the Cauchy singular integral extends to a bounded linear operator S_Γ on $L_p(\Gamma, w)$.*

In particular, $S_{\Gamma,\epsilon} \in \mathcal{L}(L_p(\Gamma, w))$ and for every $g \in L_p(\Gamma, w)$, $S_{\Gamma,\epsilon} g \to S_\Gamma g$, $\epsilon \to 0^+$, in $L_p(\Gamma, w)$ and almost everywhere on Γ.

Proof of Theorem 5.1. $(i) \Leftrightarrow (i')$: Note that

$$\int_\Gamma \left\| z^{1/2} R(z, \mathcal{A}) x \right\|^2 \frac{|\mathrm{d}z|}{|z|^{1/2}} = \int_{\mathrm{Re}\,\lambda=\sigma} \|\lambda R(\lambda^2, A)x\|^2 \frac{|\sigma^2 - \lambda^2|^{1/2}}{|\lambda|} |\mathrm{d}\lambda|.$$

Therefore, the equivalence of (i) and (i') follows simply from Theorem 2.5. Moreover, by applying Theorem 6.3, it is not difficult to show that (i') yields $\omega(C) < \sigma$, where C denotes the cosine function generated by A.

$[(i) \Leftrightarrow (i')] \Rightarrow (i'')$: Fix $\sigma > \omega(C)$. We use (i) to deduce the sectoriality of $\mathcal{A}$. The integral conditions of (i'') follow from the corresponding ones of (i') by the moment inequality, which gives

$$\|\mathcal{A}^{1/2} R(z, \mathcal{A})x\|^2 \le \tilde{c} \|R(z, \mathcal{A})x\| \|x\| + \tilde{c} \|z^{1/2} R(z, \mathcal{A})x\|^2, \qquad z \in \partial\Pi_\sigma, \ x \in X,$$

for a suitable constant $\tilde{c} > 0$.

$(i'') \Rightarrow (i')$: By Lemma 4.2, we get $z^{1/2} := |z|^{1/2} e^{i \arg z/2} \in \rho(\mathcal{A}^{1/2})$ for every $z \in \partial\Pi_\sigma$, where we fix that $\arg : \mathbb{C} \setminus \{0\} \to [-\pi, \pi)$, and

$$z^{1/2} R(z, \mathcal{A}) = \mathcal{A}^{1/2} R(z, \mathcal{A}) - R(-z^{1/2}, \mathcal{A}^{1/2}), \qquad z \in \partial\Pi_\sigma.$$

Since $\mathcal{A}^{1/2}$ is sectorial with angle less than $\frac{\pi}{2}$ and $\mathrm{Re}\,(-z^{1/2}) \le 0$ for all $z \in \partial\Pi_\sigma$, we easily get the integral condition in (i').

$(i) \Rightarrow (ii)$: Let $\sigma > \omega(C)$. First, we prove, in an alternative way to the one given in the proof of Theorem 4.1, that $\mathcal{B} := -i\mathcal{A}^{1/2}$ generates a C_0-group on X. Since $\sigma > \omega(C)$, we easily get $\sup_{\mathrm{Re}\,\lambda \ge \sigma} \|\lambda R(\lambda^2, A)\| < \infty$ and

$$\int_{\mathrm{Re}\,\lambda=\sigma} \|\lambda R(\lambda^2, A)x\|^2 |\mathrm{d}\lambda| < \infty, \qquad \int_{\mathrm{Re}\,\lambda=\sigma} \|(\lambda R(\lambda^2, A))^* x\|^2 |\mathrm{d}\lambda| < \infty, \quad x \in X.$$

As in the proof of Theorem 4.1, the moment inequality and Lemma 4.2 yield $\sup_{|\operatorname{Re}\lambda|\geq\sigma}\|R(\lambda,\mathcal{B})\|<\infty$ and

$$\int_{\operatorname{Re}\lambda=\sigma}\|R(\lambda,\pm\mathcal{B})x\|^2|\mathrm{d}\lambda|<\infty,\qquad \int_{\operatorname{Re}\lambda=\sigma}\|R(\lambda,\pm\mathcal{B})^*x\|^2|\mathrm{d}\lambda|<\infty,\quad x\in X.$$

Consequently, integration by parts and the Cauchy–Schwarz inequality show that $\mathcal{B}$ satisfies (2.8) in (ii) of Theorem 2.6. Thus, $\mathcal{B}$ generates a C_0-group, U, on X.

Moreover, e.g., Datko's theorem shows that $\|U(t)\|\leq Me^{\omega|t|}$, $t\in\mathbb{R}$, for some $0<\omega<\sigma$ and $M>0$. In particular, this gives $\frac{1}{M}e^{-(\sigma+\omega)t}\|x\|\leq\|e^{-\sigma t}U(\pm t)x\|\leq Me^{(\omega-\sigma)t}\|x\|$ for every $t\geq 0$ and $x\in X$. Applying Plancherel's theorem, we easily see that the norm $\|\cdot\|$ on X is equivalent to the following ones

$$x\mapsto\left(\int_{\operatorname{Re}\lambda=\sigma}\left\|R(i\lambda,\pm\mathcal{A}^{1/2})x\right\|^2|\mathrm{d}\lambda|\right)^{1/2},\qquad x\in X.$$

Next, a straightforward computation based on the resolvent equation shows that

$$\begin{aligned}&\frac{1}{\tilde{c}}\|R(i\lambda,\pm\mathcal{A}^{1/2})x\|\leq\|R((\sigma^2-\lambda^2)^{1/2},\pm\mathcal{A}^{1/2})x\|\leq\tilde{c}\|R(i\lambda,\pm\mathcal{A}^{1/2})x\|,\\ &\quad\operatorname{Re}\lambda=\sigma,\ x\in X,\end{aligned}$$

for a suitable constant $\tilde{c}$, where we assume now that arg takes values in $[0,2\pi)$.

Since, by Lemma 4.2,

$$R((\sigma^2-\lambda^2)^{1/2},\pm\mathcal{A}^{1/2})=[(\sigma^2-\lambda^2)^{1/2}\pm\mathcal{A}^{1/2}]R(\sigma^2-\lambda^2,\mathcal{A}),$$

it follows that

$$x\mapsto\left(\int_{\operatorname{Re}\lambda=\sigma}\left\|[\mathcal{A}^{1/2}\pm(\sigma^2-\lambda^2)^{1/2}]R(\sigma^2-\lambda^2,\mathcal{A})x\right\|^2|\mathrm{d}\lambda|\right)^{1/2},\quad x\in X,\tag{5.2}$$

are equivalent norms on X, too. Finally, elementary considerations show that (ii) holds.

$(ii)\Rightarrow(iii)$: We shall prove that the natural $\mathcal{R}_0^\infty(\Pi_\sigma)$ functional calculus for $\mathcal{A}$ is bounded, where $\mathcal{R}_0^\infty(\Pi_\sigma)$ denotes the algebra generated by the elementary rationales $(\lambda-\cdot)^{-1}$ $(\lambda\notin\overline{\Pi}_\sigma)$. Then, the boundedness of the $H^\infty(\Pi_\sigma)$ functional calculus follows from standard approximation arguments, see, e.g., [11, Section 4.2] or [28, Section 5 and Appendix F]. We follow the notation and terminology used in [28].

Fix $f\in\mathcal{R}_0^\infty(\Pi_\sigma)$. By Cauchy's theorem, we get

$$f(\mathcal{A})=\frac{1}{2\pi i}\int_\Gamma f(z)R(z,\mathcal{A})\mathrm{d}z,\tag{5.3}$$

where Γ denotes the parabola $\partial\Pi_\sigma$ oriented counterclockwise with respect to Π_σ. Fix $x\in X$. Since $[\mathcal{A}^{1/2}+u^{1/2}]R(u,\mathcal{A})$, $u\in\Gamma$, are bounded, (5.1) and (5.3) yield

$$\|f(\mathcal{A})x\|^2\leq c\int_\Gamma\left\|\frac{1}{2\pi i}\int_\Gamma f(z)[\mathcal{A}^{1/2}+u^{1/2}]R(u,\mathcal{A})R(z,\mathcal{A})x\,\mathrm{d}z\right\|^2\frac{|\mathrm{d}u|}{|u|^{1/2}}.$$

Set

$$h(u) := \frac{1}{2\pi i} \int_{\Gamma} f(z)[\mathcal{A}^{1/2} + u^{1/2}] R(u, \mathcal{A}) R(z, \mathcal{A}) x \mathrm{d}z, \quad u \in \Gamma.$$

Since $\frac{f(\cdot)}{(\cdot)-u} \in L_1(\Gamma_{u,\epsilon}, |\mathrm{d}z|)$ for every $\epsilon > 0$ and $u \in \Gamma$, the resolvent equation yields

$$h(u) = \lim_{\epsilon \to 0^+} \left[\frac{1}{2\pi i} \int_{\Gamma_{u,\epsilon}} \frac{f(z)}{z-u} \mathrm{d}z [\mathcal{A}^{1/2} + u^{1/2}] R(u, \mathcal{A}) x \right.$$
$$\left. - \frac{1}{2\pi i} \int_{\Gamma_{u,\epsilon}} \frac{f(z)[\mathcal{A}^{1/2} + u^{1/2}] R(z, \mathcal{A}) x}{z-u} \mathrm{d}z \right]. \tag{5.4}$$

We claim that

$$\lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{\Gamma_{u,\epsilon}} \frac{f(z)}{z-u} \mathrm{d}z = \frac{1}{2} f(u) \tag{5.5}$$

for every $u \in \Gamma$. Indeed, note first that it is sufficient to show this for f of the form $f(z) = \frac{1}{z-a}$, where $a \notin \overline{\Pi}_\sigma$. For every $\epsilon > 0$ and $u \in \Gamma$, set $\Gamma^{\pm}_{u,\epsilon} =: \Gamma_{u,\epsilon} \cap \{\pm \mathrm{Im}\, z > \pm \mathrm{Im}\, u\}$ and let $C_{u,\epsilon}$ denotes the arc of $\{|z-u| = \epsilon\}$ such that $\Gamma^-_{u,\epsilon} \oplus C_{u,\epsilon} \oplus \Gamma^+_{u,\epsilon} := \partial(\Pi_\sigma \cup B(u,\epsilon))$. Then, by Cauchy's theorem, we get

$$\frac{1}{2\pi i} \int_{\Gamma^-_{u,\epsilon} \oplus C_{u,\epsilon} \oplus \Gamma^+_{u,\epsilon}} \frac{f(z)}{z-u} \mathrm{d}z = f(u)$$

for $\epsilon > 0$ small enough. On the other hand, we have

$$\frac{1}{2\pi i} \int_{C_{u,\epsilon}} \frac{f(z)}{z-u} \mathrm{d}z = \frac{1}{2\pi i} f(u) \int_{C_{u,\epsilon}} \frac{1}{(z-u)} \mathrm{d}z - \frac{1}{2\pi i} f(u) \int_{C_{u,\epsilon}} \frac{1}{(z-a)} \mathrm{d}z$$
$$\to f(u)/2, \quad \epsilon \to 0^+.$$

Hence, (5.5) holds. Consequently, (5.4) shows that

$$\lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{\Gamma_{u,\epsilon}} \frac{f(z)[\mathcal{A}^{1/2} + u^{1/2}] R(z, \mathcal{A}) x}{z-u} \mathrm{d}z$$

exists for every $u \in \Gamma$. Since, by the resolvent equation,

$$\frac{1}{2\pi i} \int_{\Gamma_{u,\epsilon}} \frac{f(z) R(z, \mathcal{A}) x}{z-u} \mathrm{d}z = \frac{1}{2\pi i} \int_{\Gamma_{u,\epsilon}} \frac{f(z)}{z-u} \mathrm{d}z R(u, \mathcal{A}) x$$
$$- \frac{1}{2\pi i} \int_{\Gamma_{u,\epsilon}} f(z) R(z, \mathcal{A}) \mathrm{d}z R(u, \mathcal{A}) x,$$

(5.5) and (5.3) imply that

$$\lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{\Gamma_{u,\epsilon}} \frac{f(z) R(z, \mathcal{A}) x}{z-u} \mathrm{d}z \quad \text{and} \quad \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{\Gamma_{u,\epsilon}} \frac{f(z) \mathcal{A}^{1/2} R(z, \mathcal{A}) x}{z-u} \mathrm{d}z$$

exist for every $u \in \Gamma$. Therefore, we have

$$
\begin{aligned}
h(u) &= \frac{1}{2} f(u)[\mathcal{A}^{1/2} + u^{1/2}]R(u,\mathcal{A})x + \frac{1}{2\pi i}\int_\Gamma \frac{f(z)\mathcal{A}^{1/2}R(z,\mathcal{A})x}{z-u}\mathrm{d}z \\
&\quad + \frac{1}{2\pi i}\int_\Gamma \frac{f(z)u^{1/2}R(z,\mathcal{A})x}{z-u}\mathrm{d}z, \qquad u \in \Gamma.
\end{aligned}
$$

This leads to the following estimate

$$
\begin{aligned}
\int_\Gamma \|h(u)\|^2 \frac{|\mathrm{d}u|}{|u|^{1/2}} &\le \tilde{c}\int_\Gamma \left\| \frac{1}{2} f(u)[\mathcal{A}^{1/2} + u^{1/2}]R(u,\mathcal{A})x \right\|^2 \frac{|\mathrm{d}u|}{|u|^{1/2}} \\
&\quad + \tilde{c}\int_\Gamma \left\| \frac{1}{2\pi i}\int_\Gamma \frac{f(z)\mathcal{A}^{1/2}R(z,\mathcal{A})x}{z-u}\mathrm{d}z \right\|^2 \frac{|\mathrm{d}u|}{|u|^{1/2}} \\
&\quad + \tilde{c}\int_\Gamma \left\| \frac{1}{2\pi i}\int_\Gamma \frac{f(z)R(z,\mathcal{A})x}{z-u}\mathrm{d}z \right\|^2 |u^{1/2}|\,|\mathrm{d}u|
\end{aligned}
$$

for a suitable constant $\tilde{c}$ independent of f and x.

It is easy to check that Γ is a Carleson curve. Let $w(u) := |u|^{1/4}$, $u \in \Gamma$. Then, by [6, Theorem 2.2], the weights w and w^{-1} belong to the $A_2(\Gamma)$ class. Furthermore, by Lemma 5.2 and a standard extension procedure, the operators S_Γ and $S_{\Gamma,\epsilon}$ ($\epsilon > 0$) extend to bounded linear operators $\widetilde{S}_\Gamma$ and $\widetilde{S}_{\Gamma,\epsilon}$ ($\epsilon > 0$) on $L_2(\Gamma, w^{\pm 1}; X)$ with the preservation of the norms. Set

$$
g_1(u) := f(u)\mathcal{A}^{1/2}R(u,\mathcal{A})x \quad \text{and} \quad g_2(u) := f(u)R(u,\mathcal{A})x, \qquad u \in \Gamma.
$$

By (5.1), $g_1 \in L_2(\Gamma, w^{-1}; X)$ and $g_2 \in L_2(\Gamma, w; X)$ with

$$
\|g_1\|^2_{L_2(\Gamma,w^{-1};X)},\ \|g_2\|^2_{L_2(\Gamma,w;X)} \le b\|f\|^2_\infty \|x\|^2
$$

for a suitable constant $b > 0$. Since, by [6, Proposition 4.5], $C_0^\infty(\Gamma)$ is dense in $L_2(\Gamma, w)$, we get $\widetilde{S}_\Gamma g_i(u) = \frac{1}{2\pi i}\int_\Gamma \frac{g_i(z)}{z-u}\mathrm{d}z$ for a.e. $u \in \Gamma$, and $i = 1, 2$. Finally,

$$
\begin{aligned}
\|f(\mathcal{A})x\|^2 &\le c\int_\Gamma \|h(u)\|^2 \frac{|\mathrm{d}u|}{|u|^{1/2}} \\
&\le c'\|f\|^2_\infty\|x\|^2 + c'\int_\Gamma \|\widetilde{S}_\Gamma g_1(u)\|^2 \frac{|\mathrm{d}u|}{|u|^{1/2}} + c'\int_\Gamma \|\widetilde{S}_\Gamma g_2(u)\|^2 |u|^{1/2}|\mathrm{d}u| \\
&\le c'\|f\|^2_\infty\|x\|^2,
\end{aligned}
$$

where c' denotes an absolute constant independent of f and x.

The proof of $(iii) \Rightarrow (i)$ is standard and its details can be found, e.g., in [11, Section 4.2]. □

We conclude this section with a cosine function analogue of the Fröhlich-Weis result on dilation properties of sectorial operators with bounded H^∞ functional calculus from [22, Section 5], see also [39, Theorem 11.14].

PROPOSITION 5.3. *Let A be the generator of a cosine function C on a Hilbert space X and let $\mathcal{A} := \sigma^2 - A$ for some $\sigma > \omega(C)$. Then, $\mathcal{A}$ has a dilation to the multiplication operator*

$$\mathcal{M}f(t) = (\sigma^2 - (\sigma + it)^2)f(t), \quad t \in \mathbb{R},$$

on $L_2(\mathbb{R}; X)$, i.e., there exists an equivalent scalar product $(\cdot,\cdot)_{\mathcal{A}}$ on X, an isometric embedding $J : (X, (\cdot,\cdot)_{\mathcal{A}}) \to L_2(\mathbb{R}; X)$, such that JJ^* is an orthogonal projection from $L_2(\mathbb{R}; X)$ onto JX and

$$J^* R(z, \mathcal{M}) J = R(z, \mathcal{A}), \qquad z \in \Pi_\sigma \setminus \Pi_\omega, \tag{5.6}$$

where $\omega(C) < \omega < \sigma$.

Proof. As in the proof of Theorem 5.1$(i) \Rightarrow (ii)$, one can easily show that

$$\|x\|_{\mathcal{A}} := \left(\int_{\operatorname{Re}\lambda = \sigma} \left\| [\mathcal{A}^{1/2} + (\sigma^2 - \lambda^2)^{1/2}] R(\sigma^2 - \lambda^2, \mathcal{A}) x \right\|^2 |\mathrm{d}\lambda| \right)^{1/2}, \qquad x \in X,$$

is an equivalent (Hilbertian) norm on X. Let $J : (X, (\cdot,\cdot)_{\mathcal{A}}) \to L_2(\mathbb{R}; X) \equiv L_2(\sigma + \mathbb{R}i, |\mathrm{d}\lambda|; X)$ be given by

$$Jx(\lambda) := [\mathcal{A}^{1/2} + (\sigma^2 - \lambda^2)^{1/2}] R(\sigma^2 - \lambda^2, \mathcal{A}) x, \qquad \operatorname{Re}\lambda = \sigma, \quad x \in X.$$

Clearly, Jx has a holomorphic extension on $\{\operatorname{Re}\lambda > \omega\}$ for every $x \in X$. The verification of the statement of Proposition 5.3 follows the lines of the proof of [39, Theorem 11.14]. The main supplementary observation to be made is that $\lim_{R\to\infty} \sup_{|\phi| \le \frac{\pi}{2}} \|(Jx)(\sigma + Re^{i\phi})\| \to 0$ for every $x \in X$, which is a consequence of the fact that $(\cdot)R((\cdot)^2, A)x \in H^2(\omega; X)$, $x \in X$, and the moment inequality. □

REMARK 5.4. Note that a converse of Proposition 5.3 is also true, in the sense that the dilation property characterises cosine function generators on Hilbert spaces, see [22] for related matters for sectorial operators. For instance, the dilation equation (5.6) gives an alternative approach to the proof of Theorem 5.1$(ii) \Rightarrow (iii)$.

We refer the reader also to the series of papers [3,5,11–13,43], where the relationships between the notions of dilation, boundedness of functional calculus and numerical range are studied for general convex domains.

6. Growth bound for perturbed cosine functions

In [19], Fattorini studied the growth estimates for the cosine function C_ζ generated by the operator $A + \zeta^2$, where $\zeta \in \mathbb{C}$ and A generates a cosine function C on a Banach space X such that $\|C(t)\| \le Me^{\sigma t}$ for every $t \ge 0$ and for some constants $M, \sigma \ge 0$. In particular, he proved that for every $\zeta \in \mathbb{C}$ and every Banach space X, there exists $\widetilde{M} \ge 0$ such that

$$\|C_\zeta(t)\| \le \widetilde{M} e^{(\sigma^2 + |\zeta|^2)^{1/2} t}, \qquad t \ge 0, \tag{6.1}$$

and in the case of a Hilbert space X and a normal operator A, he showed that (6.1) can be sharpened to

$$\|C_\zeta(t)\| \le \widetilde{M} e^{(\sigma^2+|\mathrm{Re}\,\zeta|^2)^{1/2}t}, \qquad t \ge 0. \tag{6.2}$$

However, Fattorini showed that a generator A of a cosine function C for which (6.2) does not hold can be constructed for any $\sigma \ge 0$ on L_p spaces with $p \ne 2$, see [19, Example 3.3]. We refer the reader to [24] for related matters and to [20, Chapter VI] for the applications of these results to PDE.

Every generator of a bounded cosine function on a Hilbert space is similar to a self-adjoint operator, so (6.2) holds if $\sigma = 0$ (see [19, Theorem 3.1] and [18]). The question whether (6.2) holds for an arbitrary generator of a cosine function on a Hilbert space was left open by Fattorini [19, p. 240]. The following Theorem 6.1 answers Fattorini's question in the affirmative.

THEOREM 6.1. *Let A be the generator of a cosine function C on a Hilbert space X. Assume that*

$$\|C(t)\| \le M e^{\sigma t}, \qquad t \ge 0, \tag{6.3}$$

for some constants $M, \sigma > 0$. Then, for every $\zeta \in \mathbb{C}$, there exists a constant $M_\zeta > 0$ such that

$$\|C_\zeta(t)\| \le M_\zeta\, e^{(\sigma^2+|\mathrm{Re}\,\zeta|^2)^{1/2}t}, \qquad t \ge 0, \tag{6.4}$$

where C_ζ denotes the cosine function generated by $A + \zeta^2$.

To simplify notation set

$$P_\xi := \{z \in \mathbb{C} : \mathrm{Re}\, z \le \xi^2 - (\mathrm{Im}\, z)^2/4\xi^2\} = -\overline{\Pi}_\xi + \xi^2 \quad \text{and} \quad \omega_\xi := (\xi^2 + |\mathrm{Re}\,\zeta|^2)^{1/2}$$

for $\xi \ge \sigma$. At first note that, by Theorem 2.4, it is sufficient to show that the condition (ii) of Theorem 2.4 for A implies the appropriate one for $A+\zeta^2$, i.e., $\sigma(A+\zeta^2) \subset P_{\omega_\sigma}$ and

$$\sup_{\omega>\omega_\sigma} (\omega - \omega_\sigma) \int_{\mathrm{Re}\,\lambda=\omega} \|\lambda R(\lambda^2, A+\zeta^2)x\|^2 |\mathrm{d}\lambda| < \infty, \tag{6.5}$$

$$\sup_{\omega>\omega_\sigma} (\omega - \omega_\sigma) \int_{\mathrm{Re}\,\lambda=\omega} \|\lambda R(\lambda^2, A+\zeta^2)^* x\|^2 |\mathrm{d}\lambda| < \infty \tag{6.6}$$

for every $x \in X$. For the validity of the location condition for the spectrum of $A+\zeta^2$, i.e., $P_\sigma + \zeta^2 \subset P_{\omega_\sigma}$, we refer to [19, Section 1, p. 239]. For the proof of (6.5) and (6.6), we shall use the idea of the proof of the relevant result due to Fattorini, see [19, Theorem 4.2]. It is easily seen that the main tool of the proof of [19, Theorem 4.2] is a vector-valued version of the Carleson embedding theorem, which follows immediately from the classical one, see, e.g., [23, Chapter II, Theorem 3.9]. Our main task will be to prove the uniform boundedness of the Carleson constants of appropriate Borel measures on $\mathbb{C}_+$. For convenience of the reader, we provide below some technical parts of the proof.

Proof. We shall prove only (6.5), the same arguments apply to (6.6). Fix $x \in X$. Note that

$$\int_{\operatorname{Re}\lambda=\omega_\xi} \|\lambda R(\lambda^2, A+\zeta^2)x\|^2 |\mathrm{d}\lambda| = \int_{\Gamma(\xi)} \|\mu R(\mu^2, A)x\|^2 |\mathrm{d}\mu|, \quad \xi > \sigma,$$

where $\Gamma(\xi)$ is the curve parametrized in the following way:

$$\Gamma(\xi): \ \mu(\lambda) = (\lambda^2 - \zeta^2)^{1/2}, \quad \operatorname{Re}\lambda = \omega_\xi.$$

Since $P_\xi + \zeta^2 \subset P_{\omega_\xi}$, we easily get $\Gamma(\xi) \subset \{\operatorname{Re}\lambda \geq \xi\}$, $\xi > \sigma$. By the Paley–Wiener theorem, (6.3) implies that $\{\operatorname{Re}\mu > \xi\} \ni \mu \mapsto \mu R(\mu^2, A)x \in H_2(\xi; X)$ for every $\xi > \sigma$. Now, by the Carleson embedding theorem, there exists $c > 0$ such that

$$\int_{\Gamma(\xi)} \|\mu R(\mu^2, A)x\|^2 |\mathrm{d}\mu| \leq cN(\xi) \int_{\operatorname{Re}\lambda=\xi} \|\lambda R(\lambda^2, A)x\|^2 |\mathrm{d}\lambda|, \quad \xi > \sigma, \tag{6.7}$$

where $N(\xi)$, $\xi > \sigma$, is the Carleson constant of the Borel measure m_ξ on $\mathbb{C}_+$ given by

$$m_\xi(Q) := \int_{(\Gamma(\xi)-\xi)\cap Q} |\mathrm{d}\mu|$$

for all squares Q in $\mathbb{C}_+$.

We show that

$$\sup_{\sigma<\xi<\sigma+|\zeta|} N(\xi) < \infty. \tag{6.8}$$

Let $Q(s,h) := \{\lambda \in \mathbb{C} : 0 \leq \operatorname{Re}\lambda \leq h,\ s \leq \operatorname{Im}\lambda \leq s+h\}$ for every $s \in \mathbb{R}$ and $h > 0$. Note that

$$\begin{aligned}
N(\xi) &:= \sup_{h>0, s\in\mathbb{R}} \frac{1}{h} \int_{(\Gamma(\xi)-\xi)\cap Q(s,h)} |\mathrm{d}\mu| \\
&\leq \sup_{h>0, s\in\mathbb{R}} \frac{1}{h} \int_{\Gamma(\xi)\cap\{s\leq \operatorname{Im}\mu\leq s+h\}} |\mathrm{d}\mu| = \sup_{h>0, s\in\mathbb{R}} \frac{1}{h} \int_{t_\xi(s)}^{t_\xi(s+h)} \frac{|\omega_\xi + it|}{|(\omega_\xi+it)^2 - \zeta^2|^{1/2}} \mathrm{d}t \\
&\leq M \sup_{h>0, s\in\mathbb{R}} \frac{1}{h}(t_\xi(s+h) - t_\xi(s)),
\end{aligned}$$

where $M := \sup_{t\in\mathbb{R}, \sigma<\xi<\sigma+|\zeta|} \frac{|\omega_\xi+it|}{|(\omega_\xi+it)^2-\zeta^2|^{1/2}} < \infty$ and $t_\xi(s)$, $s \in \mathbb{R}$, is the solution of the following equation:

$$\operatorname{Im}\left((\omega_\xi + it)^2 - \zeta^2\right)^{1/2} = s.$$

It is not difficult to show that

$$t_\xi(s) = \frac{s^2\beta + \omega_\xi s\left(4(\omega_\xi^2 + s^2)(\omega_\xi^2 + s^2 - \alpha) - \beta^2\right)^{1/2}}{2\omega_\xi(\omega_\xi^2 + s^2)} + \frac{\beta}{2\omega_\xi}$$

for every $s \in \mathbb{R}$ and $\sigma < \xi < \sigma + |\zeta|$, where $\alpha := \operatorname{Re}\zeta^2$ and $\beta := \operatorname{Im}\zeta^2$. Since $\sup_{s\in\mathbb{R},\sigma<\xi<\sigma+|\zeta|} |t'_\xi(s)| < \infty$, it follows that (6.8) holds.

Combining (6.7) with (6.8), we get

$$\sup_{\sigma<\xi<\sigma+|\zeta|} (\xi - \sigma) \int_{\operatorname{Re}\lambda=\omega_\xi} \|\lambda R(\lambda^2, A + \zeta^2)x\|^2 |\mathrm{d}\lambda| < \infty.$$

Since $\sup_{\xi>\sigma} \frac{\omega_\xi - \omega_\sigma}{\xi-\sigma} < \infty$ and $\sup_{\omega>\sigma+|\zeta|} (\omega - \omega_\sigma) \int_0^\infty \|e^{-\omega t} C_\zeta(t)x\|^2 \mathrm{d}t < \infty$, see (6.1), we get (6.5) and the proof is complete. □

REMARK 6.2. The analysis of the proof of [19, Theorem 4.2] shows that Fattorini proved that the Carleson constants of the curves $\Gamma(\xi)\setminus\{|\operatorname{Im}\mu| \le N\}, \sigma < \xi < \sigma+|\zeta|$, are uniformly bounded for some large $N > 0$. This is sufficient for his proof. Indeed, note that under the assumptions of [19, Theorem 4.2], we have that $(\sigma + i\mathbb{R})^2 \subset \rho(A)$ and this implies that $\int_{\operatorname{Re}\lambda=\omega_\sigma, |\operatorname{Im}\lambda|\le N} \|\lambda R(\lambda^2, A+\zeta^2)x\|^2 |\mathrm{d}\lambda| < \infty$ for every $N > 0$. Our assumptions of Theorem 6.1 do not imply that $(\sigma + i\mathbb{R})^2 \subset \rho(A)$ and this is the reason why we have to be more detailed and why we complete Fattorini's considerations.

Finally, we prove the cosine function analogues of Datko's and Gearhart's theorems for C_0-semigroups, see, e.g., [2, Theorems 5.1.2 and 5.2.1].

THEOREM 6.3. *Let A be a generator of a cosine function C on a Hilbert space X. Assume that*

$$\int_0^\infty \|e^{-\sigma t} C(t)x\|^2 \mathrm{d}t \le M\|x\|^2, \qquad x \in X, \tag{6.9}$$

for some constants σ, $M > 0$. Then,

$$\omega(C) = \inf\left\{\omega \ge 0 : \sup_{\operatorname{Re}\lambda>\omega} \|\lambda R(\lambda^2, A)\| < \infty\right\} < \sigma.$$

Note that this theorem simply leads to a sharpened version of [19, Theorem 4.2]. Indeed, if (6.9) holds, then $\omega(C_\zeta) < (\sigma^2 + |\operatorname{Re}\zeta|^2)^{1/2}$ for every $\zeta \in \mathbb{C}$.

Proof. We first show that (6.9) implies

$$\sup_{t\ge0} \|e^{-\sigma t} C(t)\| < \infty. \tag{6.10}$$

Let $0 \le s \le t \le s + 1$ and $c := \sup_{t\in[0,1]} \|C(t)\|$. Then, by d'Alembert's equation, we get

$$\begin{aligned}
\|e^{-\sigma s} C(s)x\|^2 &\le 8c^2 e^{-2\sigma s} \int_s^{s+1} \left(\|C(t)x\|^2 + \|C(2t - s)x\|^2\right) \mathrm{d}t \\
&\le 8c^2 e^{2\sigma} \int_s^{s+1} \|e^{-\sigma t} C(t)x\|^2 \mathrm{d}t + 8c^2 e^{2\sigma} e^{-\sigma s} \int_s^{s+2} e^{-\sigma u} \|C(u)x\|^2 \mathrm{d}u \\
&\le 16c^2 e^{4\sigma} M\|x\|^2
\end{aligned}$$

for every $x \in X$. Therefore, (6.10) holds and, in particular, $\omega(C) \le \sigma$.

Furthermore, the Paley–Wiener theorem shows that

$$\sup_{\xi>\sigma}\int_{\operatorname{Re}\lambda=\xi}\|\lambda R(\lambda^2,A)x\|^2|\mathrm{d}\lambda| = 2\pi\int_0^\infty \|e^{-\sigma t}C(t)x\|^2\mathrm{d}t \le 2\pi M\|x\|^2, \quad x\in X. \tag{6.11}$$

Thus, $(\cdot)R((\cdot)^2, A)x \in H_2(\sigma; X)$ for every $x \in X$. Consequently, the Poisson integral representation and the Uniform Boundedness Principle yield

$$\sup_{\operatorname{Re}\lambda>\sigma}(\operatorname{Re}\lambda-\sigma)^{1/2}\|\lambda R(\lambda^2,A)\| < \infty. \tag{6.12}$$

Combining $\|R(z, A)\| \ge (\operatorname{dist}(z, \sigma(A)))^{-1}$, $z \in \rho(A)$, with (6.12), we get $\{\lambda^2 : \operatorname{Re}\lambda = \sigma\} \subset \rho(A)$ and, in particular, $\sigma^2 - A$ is sectorial. The proof of Theorem 6.1 and similar arguments as above show that

$$\sup_{\xi\ge\sigma}\int_{\operatorname{Re}\lambda=\xi}\|\lambda R(\lambda^2,A-\sigma^2)x\|^2|\mathrm{d}\lambda| = 2\pi\int_0^\infty \|e^{-\sigma t}C_\zeta(t)x\|^2\mathrm{d}t \le \tilde{c}\|x\|^2, \quad x\in X, \tag{6.13}$$

where $\tilde{c}$ is a suitable constant and C_ζ ($\zeta := i\sigma$) denotes the cosine function generated by $A - \sigma^2$. It should be remarked that one can prove (6.13) applying merely (6.11) and the resolvent equation. The moment inequality gives

$$\sup_{\xi\ge\sigma}\int_{\operatorname{Re}\lambda=\xi}\|(\sigma^2-A)^{1/2}R(\lambda^2,A-\sigma^2)x\|^2|\mathrm{d}\lambda| \le c'\|x\|^2, \quad x\in X, \tag{6.14}$$

for a suitable constant c'. Combining (6.13), (6.14), Lemma 4.2 and Theorem 4.1, we get

$$\sup_{\operatorname{Re}\lambda\ge\sigma}\int_{\operatorname{Re}\lambda=\xi}\|R(\lambda,\pm\mathcal{B})x\|^2\mathrm{d}t = 2\pi\int_0^\infty \|e^{-\sigma t}U(\pm t)x\|^2\mathrm{d}t < \infty, \quad x\in X,$$

where U denotes the C_0-group generated by $\mathcal{B} := i(\sigma^2 - A)^{1/2}$. Since $C_\zeta(t) = \frac{1}{2}(U(t)+U(-t))$, $t \ge 0$, Datko's theorem shows that $\omega(C_\zeta) < \sigma'$ for some $0 < \sigma' < \sigma$. In particular, we get

$$\{\lambda^2 : \operatorname{Re}\lambda \ge \sigma'\} \subset \rho(A-\sigma^2) \quad \text{and} \quad \sup_{\operatorname{Re}\lambda\ge\sigma'}\|\lambda R(\lambda^2, A-\sigma^2)\| < \infty. \tag{6.15}$$

Simple considerations based on (6.15) show that there exists $\sigma' < \omega < \sigma$ such that $\{\lambda^2 : \operatorname{Re}\lambda \ge \omega\} \subset \rho(A)$ and $\sup_{\operatorname{Re}\lambda\ge\omega}\|R(\lambda^2, A)\| < \infty$. Finally, combining this estimate, the resolvent equation and $\omega(C_\zeta) < \sigma'$, we easily get

$$\sup_{\xi\ge\omega}\int_{\operatorname{Re}\lambda=\xi}\|\lambda R(\lambda^2,A)x\|^2|\mathrm{d}\lambda| = 2\pi\int_0^\infty \|e^{-\omega t}C(t)x\|^2\mathrm{d}t < \infty, \quad x\in X.$$

Therefore, as in the first part of the proof, one can show that $\omega(C) \le \omega < \sigma$. $\square$

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